Forces on a sphere accelerating in a viscous fluid

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A detailed equation is proposed for the force exerted on a sphere that accelerates rectilinearly in an otherwise still fluid. In addition to the buoyant force, the fluid exerts forces that depend on (*a*) the velocity of the sphere, (*b*) the acceleration of the sphere and (*c*) the history of the motion. The equation reduces to the known theoretical solution for low velocity and large acceleration.

The proposed equation was tested and found most satisfactory for a particular case in which the velocity was not small, viz. the case of simple harmonic motion along a straight line. The acceleration (added mass) and history coefficients in the equation were evaluated experimentally. They were found to depend on the ratio of the convective acceleration to the local acceleration as measured by the parameter V^2/aD, in which V, a and D are the velocity, acceleration and diameter of the sphere, respectively. The Reynolds numbers varied from 0 to 62 during the tests.

1. Introduction

This paper presents a small part of the answer to a very old and complex problem of fluid mechanics, the problem of the dynamic force exerted by a real fluid on a submerged object if the relative velocities between the two change with time. The general situations in which both the fluid and the body move are very complicated. The motion of the fluid at a considerable distance from the body may be curved, converging and unsteady and the motion of a body relative to a fixed reference point may be very irregular. In this paper we are concerned with the rectilinear acceleration of a sphere in an otherwise quiet and viscous fluid.

Stokes (1851) investigated the simple harmonic and rectilinear oscillations of a sphere, a cylinder and an infinitely long flat plate in a viscous fluid. He omitted the convective acceleration terms in the Navier–Stokes equations and derived the expressions for forces exerted by the fluid on these objects. Each expression consists of two terms, one involving the acceleration and the other the velocity. Both terms include viscosity.

Later Basset (1888), Boussinesq (1885) and Oseen (1927) studied the rectilinear motion of a sphere which has a rapid but arbitrary acceleration in a viscous fluid. They also omitted the convective acceleration terms in the Navier–

Stokes equations in deriving their expression for force. They agree that the force on the sphere depends not only on its instantaneous velocity and acceleration, but also on an integral term which represents the effect of its entire history of acceleration. Each effect is represented by a separate term. It is important to note that the acceleration term does not include viscosity. In fact, this term is the same as the expression for force derived in an inviscid and irrotational flow.

If the force expression derived by Basset (1888), Boussinesq (1885) and Oseen (1927) is applied to an oscillating sphere and if the integral term is calculated for the oscillations after a long period of time from the start of the motion, the same expression found by Stokes (1851) can be obtained. This shows that, in a force expression valid for a specific motion, the quantity which is multiplied by acceleration does not necessarily represent the added mass as defined for an inviscid and irrotational motion.

The force expression derived by Basset (1888), Boussinesq (1885) and Oseen (1927) is valid only for a slowly moving but rapidly accelerating sphere. The authors have attempted to extend this work to include the effect of the convective acceleration terms and propose a new force expression for a rectilinear and unrestricted arbitrary motion of a sphere.

2. The problem of rectilinear motion of a sphere

The expression for force derived by Basset (1888), Boussinesq (1885) and Oseen (1927) is

$$-F = 6\pi R\mu V + \tfrac{1}{2}(\tfrac{4}{3}\pi R^3)\rho a + 6R^2(\pi\mu\rho)^{\frac{1}{2}}\int_0^t \frac{a(t')}{(t-t')^{\frac{1}{2}}}dt'. \tag{1}$$

in which μ is the viscosity of the fluid and V, a and R are the velocity, acceleration and radius of the sphere, respectively. The first term on the right will be denoted as $-F_V$. It is equal and opposite to the steady-state viscous drag on the sphere. The second term is $-F_A$ and has the same magnitude as the resistance of an accelerating sphere in irrotational motion. The third term will be denoted as $-F_H$, the effect of the history of acceleration. Since the convective acceleration terms are disregarded, the motion of the sphere is confined to rapid accelerations and low velocities.

Now (1) will be extended to derive a new equation applicable to situations in which the convective acceleration is important. Let us consider the harmonic oscillation of a sphere in an otherwise still fluid (figure 1).

The sphere starts from rest at location 1 where $t = 0$. An instant later the velocity is almost zero and the local accelerations of the fluid particles predominate. At this time the force on the sphere can be expressed in added mass form as

$$-F_A = C_A \tfrac{4}{3}\pi R^3 \rho a, \tag{2}$$

in which C_A is the added-mass coefficient. C_A is to be determined by experiment, and it may be a function of dimensionless combinations derived by dimensional analysis.

Next consider location 2, where the velocity is a maximum. Since the acceleration is zero, the force according to (2) is zero for finite values of C_A. But experiments show that the force is not zero. According to dimensional reasoning, it can be written as

$$-F_V = \tfrac{1}{2}C_V \pi R^2 \rho |V| V, \tag{3}$$

in which C_V is the velocity or drag coefficient. C_V also needs to be determined experimentally as a function of pertinent dimensionless parameters.

Now one more step will be taken. Suppose that the sphere started at location 1, completed its cycle, came back to location 1, and started its new cycle. It will be noticed that at this time the force acting on the sphere is different from the force recorded initially, although the velocities and accelerations are the same in both instances. This shows that a third term showing the effect of the history

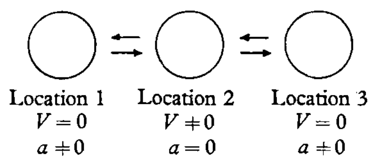

FIGURE 1. Simple harmonic motion.

of the motion must be included in the expression for the force. The need for a third term also is evident in case of a rapid stop. In this case both the acceleration and velocity will be zero, but some force will act on the sphere due to the residual velocity of the fluid. This force will diminish rapidly as time elapses, of course, but its presence shows that some history term is required.

The history force $-F_H$ may be expressed by multiplying the last term of (1) by an undetermined coefficient C_H, whose value may depend on dimensionless combinations involving acceleration, velocity and time. If this is done and the three forces are added the result is

$$-F = \tfrac{1}{2}C_V \pi R^2 \rho |V| V + C_A \tfrac{4}{3}\pi R^3 \rho a + C_H R^2 (\pi \mu \rho)^{\frac{1}{2}} \int_0^t \frac{a(t')}{(t-t')^{\frac{1}{2}}} dt'. \tag{4}$$

Let us now compare equations (4) and (1) term by term. If the coefficients in (4) are to be correct when the acceleration is large and the velocity small, the terms must be identical to those in (1). For this limiting case, the first term on the right of (1) is the steady-state drag, and hence the first term on the right of (1) must represent the same thing. Moreover, if, after some arbitrary motion, the acceleration remains zero for a long time, (4) must reduce to the steady-state drag no matter what the velocity. Both these requirements are met if we choose C_V in (4) as the steady-state drag coefficient for a sphere.

But the question arises as to whether or not such a choice can be made. Suppose, for example, that in some sort of oscillatory motion the acceleration and the acceleration integral terms both were zero at a given instant but the velocity were not. Then the velocity coefficient could not be chosen; it would be fixed by the instantaneous force (as measured experimentally) and velocity. On the other hand, for simple harmonic motion, the integral and the acceleration do not go to zero simultaneously and C_V may be chosen to fit the previously mentioned requirements. Hence, it will be assumed in what follows that C_V is the well-known steady-state drag coefficient, a function of the Reynolds number computed from the instantaneous velocity.

Continuing with the comparison of (4) and (1) it can be seen next that the coefficient C_A must equal $\frac{1}{2}$ where the velocity is small compared to the acceleration. As stated before, this is the value of the added-mass coefficient for irrotational motion, and one would expect it to be correct at least at the beginning of motion in a real fluid.

Finally, it must be noted that C_H should equal 6 for large accelerations and small velocities. Having chosen C_{V}, neither C_A or C_H is arbitrary but must be determined by experiment.

Since (1) does not permit convective acceleration to be significant, it is certain that the coefficients C_A and C_H in (4) will depend on its magnitude. To form dimensionless parameters that involve convective acceleration and are physically significant, one may write force ratios for a unit cube as follows:

$$\frac{F_{\text{conv. acc.}}}{F_{\text{friction}}} = \frac{\rho u(\partial u/\partial x)}{\mu(\partial^2 u/\partial y^2)} \simeq \frac{\rho(V^2/D)}{\mu(V/D^2)} = \frac{\rho VD}{\mu}, \tag{5a}$$

$$\frac{F_{\text{conv. acc.}}}{F_{\text{local acc.}}} = \frac{\rho u(\partial u/\partial x)}{\rho(\partial u/\partial t)} \sim \frac{\rho(V^2/D)}{\rho a} = \frac{V^2}{aD}, \tag{5b}$$

where D is the diameter of the sphere.

The first ratio is the well-known Reynolds number Re. If the forces due to convective acceleration are small compared to the shear stress, the Reynolds number will be small. For a given viscosity, a small Reynolds number can be obtained with either a small velocity or a small diameter of sphere. Experiments made for steady-state conditions indicate that, if $Re < 1$, the contribution of the convective acceleration is not significant.

The second ratio will be called the acceleration number and denoted as Ac. This number was used by Iversen & Balent (1951) and Keim (1956), who derived it by dimensional reasoning only.

If the forces due to convective acceleration are small compared to the forces due to local acceleration, the acceleration number will be small. A small acceleration number can be obtained if the velocity of the sphere is small, its acceleration is high and its diameter is large. There should be a limit of the acceleration number below which the contribution of the convective acceleration can be disregarded.

Thus, if the density and the viscosity of the fluid and the motion of the sphere are given, there may be a range in which (1) can be applied. In this range $Re < 1$ and the acceleration number is below a certain limit to be determined.

The variations of the coefficients C_A and C_H with the Reynolds number and the acceleration number have been determined by the first author for 69 different simple harmonic motions. The values of C_A and C_H were determined at $\omega t = \frac{3}{4}\pi$ and $\omega t = \frac{1}{4}\pi$ and $\frac{1}{2}\pi$, respectively, in which ω is the angular frequency. These values were used along with the published values of C_V to calculate the forces acting on the sphere over the entire cycle of the motion. The calculated forces agree very well with the measured forces. Before going into the details of the tests, we shall describe briefly the equipment used to produce the simple harmonic motions.

3. Equipment and tests

A sketch of the equipment used to impart simple harmonic motion to a sphere in a tank full of oil is shown in figure 2.

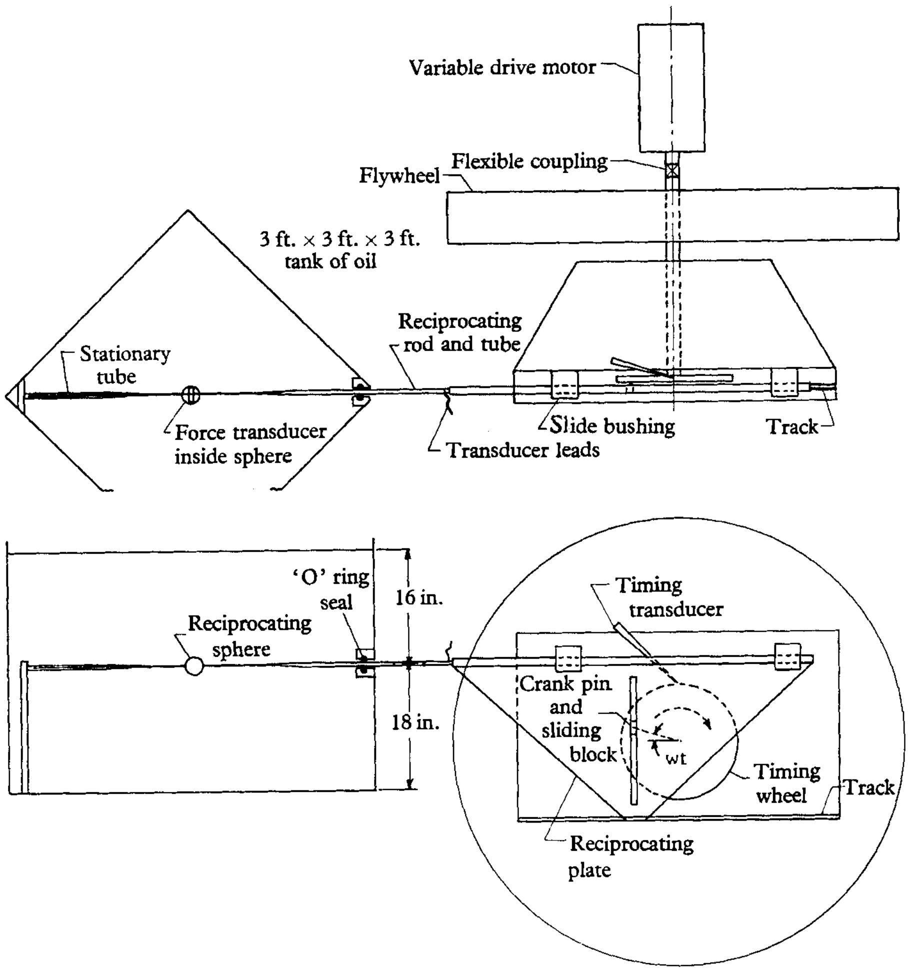

FIGURE 2. Schematic diagram of apparatus for measurement of force on reciprocating sphere.

As the motor turns the flywheel and timing wheel, the plate, rod and sphere reciprocate. The speed of the flywheel can be set between 17 and 159 r.p.m. The flywheel serves to dampen the vibrations produced by the variable drive. The sliding block is attached to the timing wheel with a crank pin, and the location of the pin can be shifted along the radius of the wheel. Thus the amplitude of the harmonic motion, A_0, can be varied. By combining different speeds and amplitudes, it is possible to produce a large number of different simple harmonic motions.

The force exerted on the sphere and the speed of the motor were measured by transducers utilizing strain gauges. The amplifier and the recording instrument used were very sensitive. For example, heavy footsteps near the equipment could disturb the measurements, although the floor was concrete. Disturbances caused by machinery in other parts of the building or by passing trucks were noticed by the authors, but the amplitude of these disturbances was very small. The data were obtained when all other equipment in the laboratory was quiet.

The amplifier and the recording instrument were manufactured by the Consolidated Electrodynamics Corporation. The technique for recording is based on the rotation of a loop of wire placed in a magnetic field. A change in the current passing through this wire causes the loop to rotate, and light beam reflected from a mirror attached to the loop travels across photographic paper which is moving at a certain speed. In this manner, the variation of current passing through the loop is recorded. The loop can oscillate at frequencies up to 3000 c/s.

A second light beam is projected on the photographic paper through a synchronized shutter at 0·1 sec intervals. This light beam produces transverse lines on the photographic paper. The distance measured between ten of these lines gives the average speed of the photographic paper accurately. The speed of travel of the photographic paper can be set to a selected value by means of a gear system. The paper is developed just after exposure.

The speed of the timing wheel was measured as follows. Two strain gauges were cemented to the upper surface of a cantilever beam and two to the lower surface. They provided four active arms of a Wheatstone bridge and permitted measurement of tensional and compressive forces acting on the lever. One end of this beam touched the periphery of the timing wheel and was pressed against it. The bridge was balanced under this pressure. On the periphery of the timing wheel, recesses were made at intervals of $\frac{1}{8}\pi$ rad. As the recesses went by, the end of the beam dropped into them and the change in strain caused an unbalanced current in the Wheatstone bridge. This unbalanced current was amplified and sent to another loop of wire placed in the magnetic field. As a result, a third light beam made a mark on the photographic paper sixteen times per revolution. Thus, since the rate of travel of the photographic paper was known, the speed of the timing wheel also could be determined.

The transducer for measuring the forces exerted on the sphere was located inside the sphere and consisted of a small brass frame with strain gauges glued on it.

As shown in figure 3 the sphere was glued to the bases of the frame and it was not in contact with the reciprocating tube. In effect the two vertical members of the frame were pinned to the tube at their centres. Since the tube passed through these members, most of their width was occupied by the hole, and the result was practically a centre hinge in each. Thus they behaved like four cantilever beams.

Again, the strain gauges constituted the four active resistances of a Wheatstone bridge. The bridge was balanced when the sphere was not in motion. When it moved, the beams deformed, the resistances of the strain gauges changed, and the balance of the bridge was disturbed. The resulting current, which represented the force exerted by the rod on the sphere and its contents, was amplified and recorded on the photographic paper as previously explained.

The transverse displacement of the trace of the light beam on the photographic paper was calibrated in units of force, using 5 g intervals. Within these intervals the variation of displacement with force was assumed to be linear. When the amplifier was set at attenuation *I*, its highest sensitivity, 5 g caused a displacement of 0·09 in.

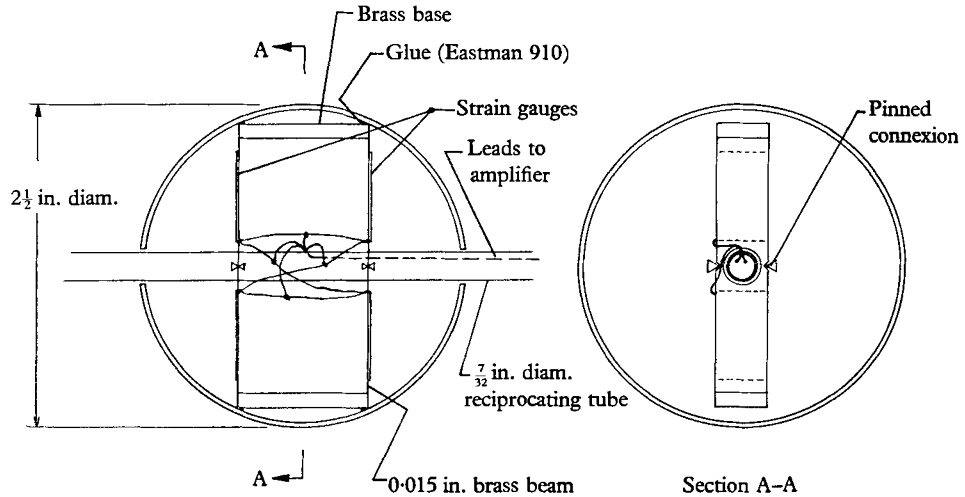

FIGURE 3. Force transducer.

The force transducer was not suitable for measuring rapid changes in force because of its low natural frequency—30 to 40 c/s—but it was satisfactory for the tests conducted. The low frequency stemmed from the need to measure small forces accurately with ordinary strain gauges and the comparatively large mass of the oil-filled sphere. To get the required sensitivity the brass cantilevers were made only 0·015 in. thick.

At points of the cycle where the force exerted on the reciprocating plate by the block and crankpin reversed, a small knock was practically unavoidable because of the slight clearance in the crankpin and main shaft bearings and the clearance between the block and the sides of the slot in the plate. Because of the knock, the sphere started to oscillate at its natural frequency twice during each cycle. This, of course superimposed unwanted vibrations on the photographic record of the periodic force being measured. Considerable care in manufacturing the slot and fitting the block to it was required to prevent the vibrations from interfering with the accuracy of the tests. However, for most runs some vibrations were unavoidable and a smooth curve was drawn through the mid-points of these recorded vibrations. A typical example is shown in figure 4.

The photographic record was measured. Tables of force *vs* time for the various runs were constructed. The scale of the record was such that a human error in measurement of 0·01 in. in the transverse direction caused an error in force of from $\frac{1}{2}$ g depending on the attenuation used in the amplifier. Similarly, an error of 0·01 in. in the longitudinal direction would cause up to 5 g error in force when the speed was high. As may be seen from tables 1 to 5† the measured forces varied from about -250 g to $+250$ g. Hence, it was necessary to scale the photographic record with great care to obtain reasonable accuracy. Because

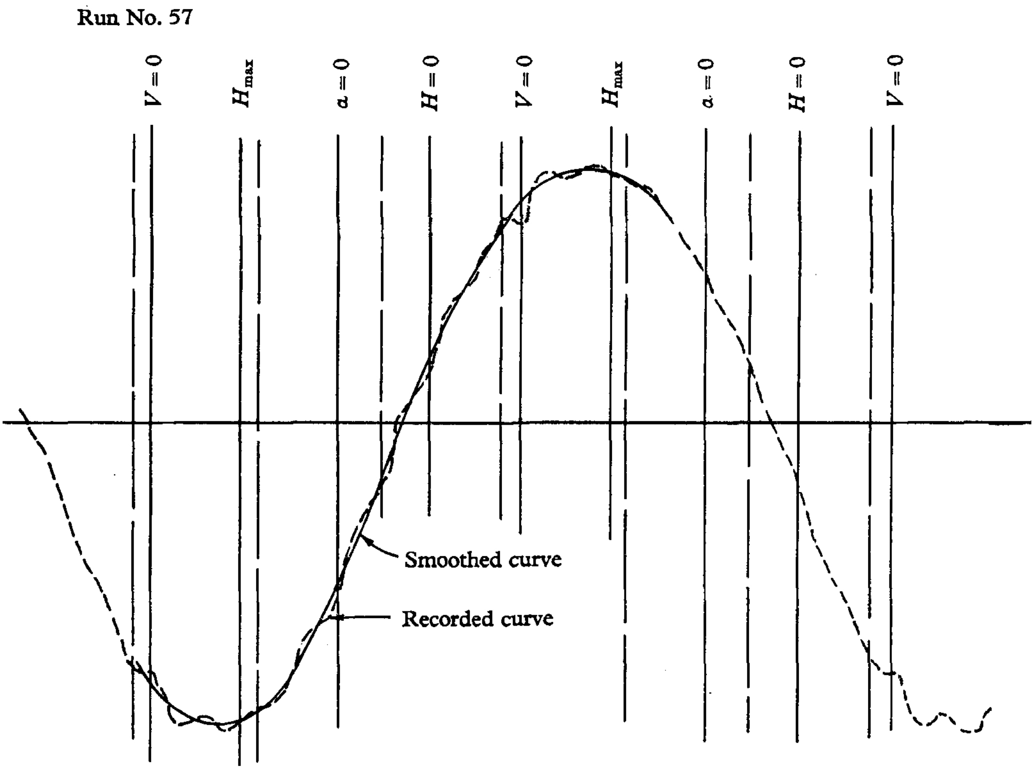

FIGURE 4. Force on reciprocating sphere.

the attenuation of the amplifier and the paper speed were adjustable, the large errors occurred for runs in which the peak force were large. Thus, judging from the consistency of the results, a satisfactory percentage accuracy was achieved.

The viscosity and the density of the oil was $1{\cdot}65 \times 10^{-2}$ lb. sec/ft.^{2} and 1·725 slug/ft.^{3}, respectively. The tests were conducted for amplitudes $A_0 = 1$, 2, 3 and 4 in. and for a wide range of speeds. A run was made by selecting a certain A_0 and setting the speed of the motor to a certain value. During the tests the Reynolds numbers varied from 0 to 62. The acceleration number varied from 0 to ∞ during each cycle.

The force measured by the transducer was that required to accelerate the mass of the sphere and the oil and brass parts inside it plus the force exerted on

† The detailed tables of data are being held in the Editors' files and will be loaned to the reader on request. The same tables are also included in Research Report 128 (in preparation), U.S. Army Cold Regions Research and Engineering Laboratory, Hanover, N.H., U.S.A. and in the Ph.D. dissertation of Odar (1962).

the sphere by the fluid external to it. In order to find the effective accelerated mass, the sphere was filled with oil, reciprocated in air, and the force recorded. Since the clearance between the shell of the sphere and the reciprocating tube was very small, there was no noticeable oil leakage. The shell of the sphere was translucent and any free surface inside the sphere could be observed. Using the force record and the known accelerations that went with it, the effective mass of the sphere and its contents was calculated to be 148 g. This figure was checked roughly by holding the tube in the vertical position. In this position, however, the oil started to leak immediately and the weight was down to 145 g by the time a measurement could be made.

4. Evaluation of the coefficients

The motion of the sphere can be expressed as

$$x = A_0 \cos \omega t, \tag{6a}$$

$$V = -A_0 \omega \sin \omega t, \tag{6b}$$

$$a = -A_0 \omega^2 \cos \omega t. \tag{6c}$$

If these quantities are introduced in (4)

$$F = C_V \pi R^2 \tfrac{1}{2} \rho A_0^2 \omega^2 |\sin \omega t| \sin \omega t + C_A \tfrac{4}{3} \pi R^3 \rho A_0 \omega^2 \cos \omega t + C_H R^2 (\pi \rho \mu)^{\frac{1}{2}} \int_{t_0}^{t} \frac{A_0 \omega^2 \cos \omega t'}{(t-t')^{\frac{1}{2}}} dt' \tag{7}$$

is obtained. The integral's lower limit, t_0, is the time at which the motion starts. For convenience, the oscillations after a long period of time from the start of the motion are considered. Thus, the lower limit of the integral in (7) is changed to $-\infty$. The value of the integral becomes

$$A_0 \omega (\tfrac{1}{2} \pi \omega)^{\frac{1}{2}} (\cos \omega t + \sin \omega t).$$

The forces due to the acceleration, velocity, history and inertia are denoted by F_A, F_V, F_H and F_I, respectively. Thus

$$F_A = C_A \tfrac{4}{3} \pi R^3 \rho A_0 \omega^2 \cos \omega t, \tag{8a}$$

$$F_V = C_V \pi R^2 \tfrac{1}{2} \rho \omega^2 A_0^2 |\sin \omega t| \sin \omega t, \tag{8b}$$

$$F_H = C_H \pi R^2 (\tfrac{1}{2} \mu \rho)^{\frac{1}{2}} A_0 \omega^{\frac{3}{2}} (\sin \omega t + \cos \omega t), \tag{8c}$$

$$F_I = M_{\text{eff}} A_0 \omega^2 \cos \omega t. \tag{8d}$$

As stated previously, C_V is assumed to be the steady-state drag coefficient dependent on the Reynolds number calculated by using instantaneous velocity. In what follows, values of C_V will be taken from Lapple (1951), table 1.

In order to find the variation of C_A with the Reynolds number or the acceleration number, or both, the location where $F_H = 0$ is chosen. According to (8c), $\omega t = \frac{3}{4}\pi$ where $F_H = 0$. At this location the force measured by the transducer is

$$F = F_A + F_V + F_I.$$

Hence, calculating F_V and F_I, and using the known total force at location $\omega t = \frac{3}{4}\pi$, F_A and consequently C_A can be evaluated. This was done. The tests showed that

C_A changes with the acceleration number only. The variation of C_A with the acceleration number is shown in figure 5. Each of the four plotted points is the average C_A calculated from at least ten runs at different values of the Reynolds number.

Using these plotted values of C_A, the total force at location $\omega t = \frac{3}{4}\pi$ was then calculated for all runs, as shown in table 1,† to check the accuracy of the C_A values and to illustrate that they do not depend on Re. The consistency of the data and their independence of Re may be judged by comparing the total force so calculated with the measured force (last two columns). A measured force represents the average of two measurements at an interval of π. It is striking that, although the values of C_V used in the C_A calculations are functions of the Reynolds number, the values of C_A themselves do not depend on Re.

The next step is to evaluate C_H, and a logical beginning is to calculate it for points in the cycle where the acceleration is zero and the history force consequently is the measured force F minus F_V. The acceleration is zero at $\omega t = \frac{1}{2}\pi$, and calculations at this point for a number of runs gave a consistent value of $C_H = 2{\cdot}88$. The value did not depend on Re, but possible dependence on Ac could not be determined since Ac is infinite for all occasions when the acceleration is zero. A comparison of measured and calculated forces using $C_H = 2{\cdot}88$ is shown for all the runs in table 2.†

Because C_V has been chosen, we know the value of the sum $F_A + F_H$ at all points of a cycle. We have so far computed C_A at $\frac{3}{4}\pi$ and C_H at $\frac{1}{2}\pi$. But now it is necessary to make a logical decision as to how to divide the sum $F_A + F_H$ into separate parts for locations on the cycle where neither is clearly zero. Since C_A already is known as a function of the acceleration number at the location $\omega t = \frac{3}{4}\pi$, it is most convenient to make these values of C_A (shown on figure 5) hold throughout the cycle. If this is done, C_H becomes determinate at all points. Then the only remaining question is whether or not C_H, computed on this basis, will show a unique dependence on Re and Ac for different runs and different locations on the cycle.

To investigate this question, a place on the cycle where the absolute value of the term $(\cos\omega t + \sin\omega t)$ is a maximum was chosen, namely, $\omega t = \frac{1}{4}\pi$. Then F_V, F_A and F_I were calculated according to equations (8a, b, d) and, using the measured force from a variety of runs, C_H was determined from (8c). It was found that C_H changed with the acceleration number only. The variation of C_H with the acceleration number is shown in figure 5. Each of the four points plotted is the average C_H calculated from at least ten runs at different values of Re.

Using these plotted values of C_H and the values of C_A in figure 5, the total force at location $\omega t = \frac{1}{4}\pi$ was then calculated for all the runs, as shown in table 3,† to check the accuracy of the C_H values and to illustrate that they do not depend on Re. The consistency of the data and the independence of Re may be judged by comparing the total force as calculated with the measured force (last two columns).

Next, tentative values of the total force at intervals of $\frac{1}{8}\pi$ were calculated to see if the values of C_H established at $\omega t = \frac{1}{4}\pi$ by the procedure just described were adequate. These values are compared with measured forces in table 4.† The

† See footnote p. 309.

agreement is excellent, even though some values of C_A and C_H were taken from the dashed portions of the curves. Figure 6 shows an example of the agreement.

The close match between the tentative computed values of force and the measured values shows that the choice of C_A was a fortunate one and answers the question of whether C_H is unique. In retrospect we now understand that,

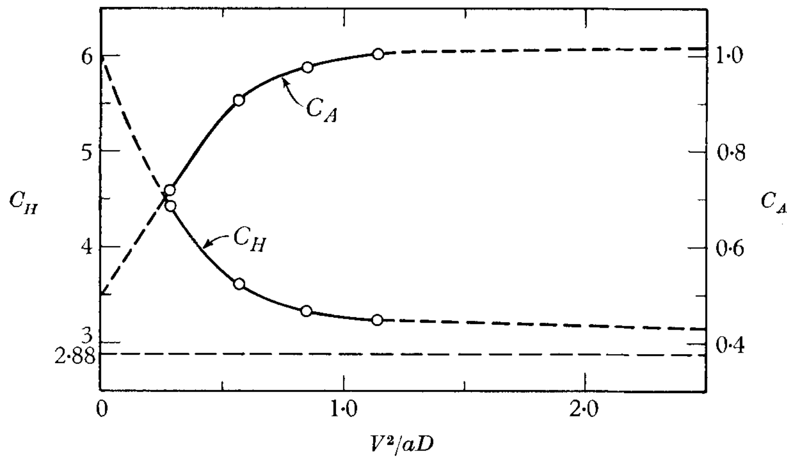

FIGURE 5. Variation of C_A and C_H with $Ac = V^2/aD$.

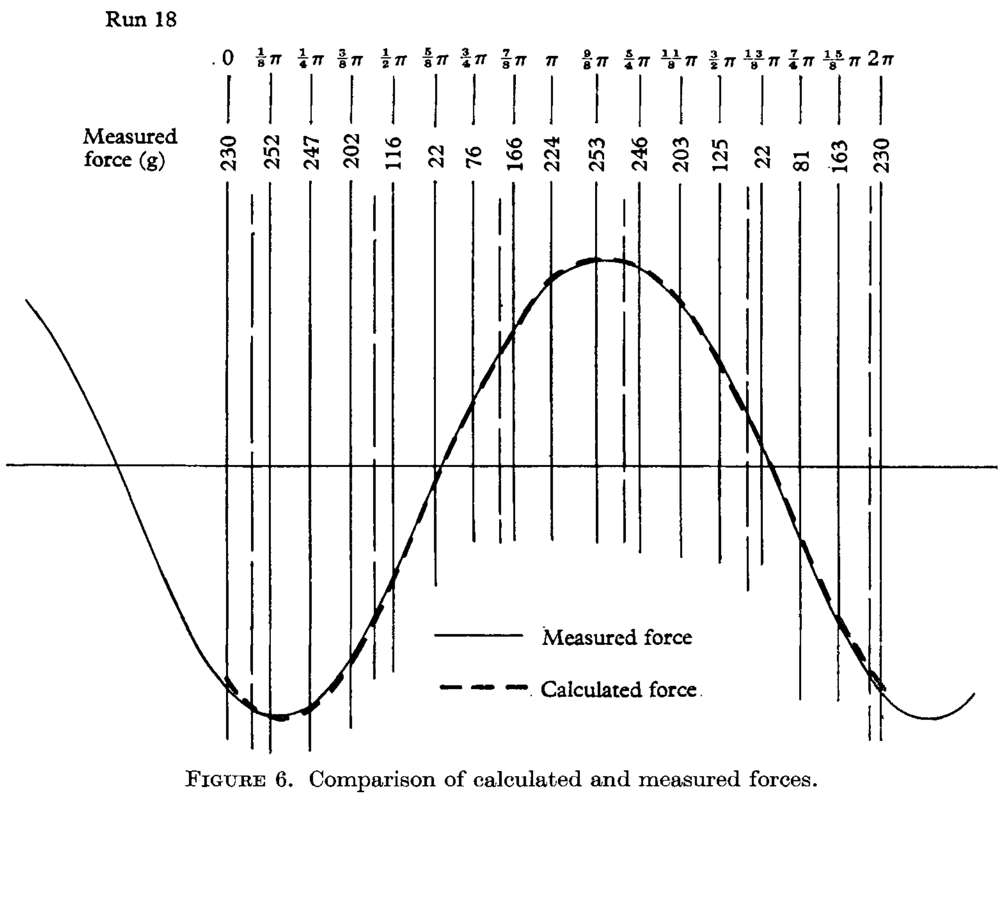

FIGURE 6. Comparison of calculated and measured forces.

if the following procedure is adopted, we get values of C_V, C_A and C_H that allow us to calculate the force exerted by the oil on the sphere:

(A) Choose C_V as the usual steady-state function of the Reynolds number.
(B) Calculate C_A at $\omega t = \frac{3}{4}\pi$, where F_H is zero.
(C) Apply the C_A from (B) for the whole cycle.
(D) Use choices (A) and (C) to calculate C_H.

The procedure is known to be valid only for the range of the present tests, of course.

A comment on the values of C_A and C_H when V and consequently, V^2/aD are zero, is in order. When V is zero, F_V is zero and the measured force is the sum

$$F_A + F_H + F_I.$$

Even though the velocity of the sphere is zero, one cannot be certain that (1) will apply, because residual motion of the liquid may produce convective accelerations that have an appreciable effect on force. The question of whether or not it does apply was answered affirmatively as follows. Using values of C_A and C_H of $\frac{1}{2}$ and 6, respectively, values of the total force at $\omega t = 2\pi$ were computed for all the runs. A comparison of the computed and measured forces is given in table 5.† The two agree quite well.

For this reason the zero values of the coefficients in figure 5 were plotted as $\frac{1}{2}$ and 6.

5. Conclusions

Neglecting buoyancy, the force exerted by a fluid on a smooth sphere which performs steady, rectilinear, harmonic oscillations in the fluid is given by

$$F = C_V \pi R^2 \tfrac{1}{2}\rho A_0^2 \omega^2 |\sin \omega t| \sin \omega t + C_A \tfrac{4}{3}\pi R^3 \rho A_0 \omega^2 \cos \omega t + C_H \pi R^2 (\tfrac{1}{2}\mu\rho)^{\frac{1}{2}} A_0 \omega^{\frac{3}{2}} (\cos \omega t + \sin \omega t). \quad (9)$$

The equation applies at least up to a Reynolds number $2\rho A_0 \omega R/\mu$ of 62.

The coefficients C_A and C_H are independent of the Reynolds number, but depend physically on the importance of the convective acceleration compared to the local acceleration, i.e. upon the ratio V^2/aD, as shown in figure 5. Values of C_A and C_H are experimental except at $V^2/aD = 0$. Here they are the theoretical values $\frac{1}{2}$ and 6, respectively, from (1) which is obtained by solving Navier–Stokes equations for the conditions where the convective acceleration terms can be disregarded. The coefficient C_V is the well-known drag coefficient for steady translation of a smooth sphere, a function of the Reynolds number.

Equation (9) is a particular form of an equation devised by the authors to express the force on a sphere having arbitrary straight-line motion, namely,

$$-F = \tfrac{1}{2} C_V \pi R^2 \rho |V| V + C_A \tfrac{4}{3}\pi R^3 \rho a + C_H R^2 (\pi\rho\mu)^{\frac{1}{2}} \int_0^t \frac{a(t')}{(t-t')^{\frac{1}{2}}} dt'. \quad (4)$$

Although it has a rational basis in (1)

$$-F = 6\pi R \mu V + \tfrac{1}{2}(\tfrac{4}{3}\pi R^3) \rho a + 6R^2 (\pi\rho\mu)^{\frac{1}{2}} \int_0^t \frac{a(t')}{(t-t')^{\frac{1}{2}}} dt' \quad (1)$$

† See footnote p. 309.

and is most satisfactory for simple harmonic motion, (4) needs to be tested for other rectilinear motions. In particular, it is desirable to find out if the values of C_A and C_H determined from the present tests are generally useful.

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